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Mandelbrot Cascade Measures Independent of Branching Parameter

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Mandelbrot cascade measures were introduced to explain intermittency in fully developed turbulence. They are defined by the scale hierarchy with a fixed branching parameter c and by the distribution of breakdown coefficients which are responsible for the transport of energy from larger to smaller scales. We show that the measures corresponding to both conservative and nonconservative cascades strongly depend on the parameter c. In particular, only Lebesgue measure can be generated by a cascade process with an arbitrary integer c.

KEY WORDS: Cascades; multifractals.

1. INTRODUCTION

Fully developed turbulence has the property of intermittency.(1) Two approaches have been suggested to explain this phenomenon. The earlier one involves multiplicative cascades(2–4) which implement Richardson's idea of energy being transported in turbulence from larger to smaller scales in the inertial range. The other approach is based on the postulate that the field of local energy dissipation is multifractal,(1) i.e., there exists an hierarchy of subsets S_α of fractional dimension $f(\alpha)$ where the field has a Holder exponent α. Both approaches ultimately explain nonlinearity of the scaling exponents $\tau(q)$ for the structure functions of the dissipation field ε:

$$\left\langle \left[\int_{\Delta} \varepsilon(x)\, dx \right]^q \right\rangle \sim |\Delta|^{\tau(q)+1}$$

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where $\langle \cdot \rangle$ denotes space averaging. (Here we consider a 1-D space.) A relation between these two approaches can be established using multifractal formalism (for rigorous results see ref. 5).

Mandelbrot gave a mathematically rigorous formalization of the multiplicative cascade.[4] His treatment relies on two assumptions which look reasonable as a first approximation: the ratio c of adjacent scales is constant (c is an integer greater than one) and the Kolmogorov diffusion scale δ is equal to zero.

A cascade is realized as follows: the energy of an eddy of scale L_k is transported to c eddies of the next scale L_{k+1} with random coefficients $\{w_i, i=1,..., c\}$, $E(\sum w_i)=1$. The operation is repeated in an iterative manner, the coefficients $w_i(L)$ corresponding to different eddies that are independent and identically distributed. The case $\sum w_i \equiv 1$ corresponds to conservative cascades; it occurs in 3-D turbulence.[6] The case of independent and identically distributed weights w_i, $Ew_i = 1/c$ is usually referred to as the Mandelbrot cascade. The above iterative process converges to a nontrivial cascade measure $\mu(dx)$ if $Ew_* \log_c w_* < 1$ and $P(w_* > 0) = 1$, where $w_* = w_{i_*}$ and i_* is a random index taking the values $1,..., c$ with equal probabilities (see refs. 7 and 8). The measure μ is regarded as an idealized model for the local dissipation field in turbulence at large Reynolds numbers.

The question as to how the parameter c should be chosen is debatable. Denoting the inertial scale range by (l_0, L_0), the values l_0, L_0 and c are connected through the relation $L_0/l_0 = c^N$ with an integer N. Sreenivasan and Stolovitzky[6] suggest the value $c=2$, since the Navier–Stokes equation involves a nonlinearity of the second order. However, other researchers[9–11] treat discreteness in cascade dynamics as a mere necessity in order to be able to describe a physical object involving a continuous set of scales. (It is supposed that the left-hand side of $L_0/l_0 = c^N$ can be replaced by any scale (l_1, L_1) from the interval (l_0, L_0)). No mathematical construction to realize the second standpoint has been provided. For this reason, when one proceeds from the second standpoint, one will naturally pose the following question within the framework of the already available theory: do cascade measures exist that are independent of the parameter c? This means that a cascade measure can be realized by a cascade process with an arbitrary parameter $c = 2, 3,\dots$. The answer is negative; more precisely, if a cascade measure can be realized with parameters $c = 2, 3, 5$ then it is the Lebesgue measure. This question for Mandelbrot cascades was considered in ref. 12.

2. SCALE INVARIANT TAU FUNCTIONS

The scaling exponents $\tau(q)$ for a cascade measure μ are defined as follows:

$$\tau(q) = \lim_{n \to \infty} \frac{\log \sum_i' \mu(\Delta_{in})^q}{\log \Delta_n} \tag{1}$$

where intervals Δ_{in} of length $\Delta_n = c^{-n}$ correspond to a c-adic division of the original space and the summation $\sum'$ involves nonzero elements $\mu(\Delta)$ only. There is a naive way of calulating $\tau(q)$: replace the space averaging $\langle \cdot \rangle$ by ensemble averaging (the mathematical expectation E) in the relation $\sum' \mu(\Delta_{in})^q = \langle \mu^q(\Delta_{in}) \rangle \Delta_n^{-1}$, and the measure μ by the pre-limiting measure μ_n, $n \gg 1$, where n is the number of iterations. This method yields a τ-function of the form

$$\tilde{\tau}(q) = q - \log_c Ew_*^q - 1 \tag{2}$$

for all q for which the moments exist. The rigorous result in ref. 5 is different:

$$\tau(q) = \begin{cases} \tilde{\tau}(q) & q_- < q < q_+ \\ a_\pm q & q/q_\pm > 1 \end{cases} \tag{3}$$

where the lines $y = a_\pm q$ are tangent to the concave curve $\tilde{\tau}(q)$ at the points $1 \leqslant q_+ \leqslant \infty$ and $0 \geqslant q_- \geqslant -\infty$. According to ref. 13, the limit in (3) is to be understood in the ordinary sense with probability one.

The exact result (3) leads to a monotone dependence of the scaling coefficients on q and to nonnegative multifractal dimensions, i.e., the Legendre transform of τ is positive. Consequently, the Novikov inequalities for $\tau(q)$ in turbulence theory (see ref. 1) are automatically valid, while criticisms of Kolmogorov's lognormal hypothesis lose their main support (see ref. 12 where this feature was treated rigorously for the first time). Furthermore, the empirical data on $\tau(q)$, $q = p/3$, $p = 1,..., 18$[14] are in perfect agreement with theoretical calculations based on (2, 3) with a lognormal variable w_*. The agreement is due to the linear correction (3) of the $\tilde{\tau}$-function taken from ref. 14. The lognormal model thus remains sufficient for practical descriptions of empirical data on $\tau(q)$; this model contains only one parameter $\mu_0 = 1 - \tilde{\tau}(2)$.

As one can see, the empirical data relating to scaling exponents are easily fitted with a model τ. This may indicate a possibility to use a comparatively narrow class of functions for parameterization of $\tau(q)$. Since c is not known, it is natural to restrict ourselves to consideration of those $\tau(q)$ only which are independent of c. That means that the τ-function can be derived from a cascade generator (w_i, $i = 1,..., c$) of arbitrary dimension $c = 2, 3,...$. A description of this class of τ-functions is a partial solution to the problem posed in the present note. In virtue of (2), (3) one needs to describe functions of moments Ew_*^q, $Ew_* = 1$ which are representable in the form

$$Ew_*^q = (E[w_*(c)]^q)^t, \qquad t = \ln 2/\ln c, \quad c = 3, 4, \ldots \tag{4}$$

where $w_*(c)$ is a quantity that is related to a cascade generator of dimension c; $w_* = w_*(2)$. I am unaware whether a full description of the class of such functions of moments is available. However, when one considers a subclass of this for which the representation (4) is valid for all t (it is sufficient, if this is so for $t = 2, 3, \ldots$ or formally $c = 2^{1/n}$), then one arrives at multiplicative infinitely divisible (MID) random variables w_*. A random variable w_* of this type admits (by definition) the multiplicative representation $w_* = \xi_1, \ldots, \xi_n$, $\xi_i \geqslant 0$ with independent, identically distributed factors for any n. The MID class first appeared in refs. 9 and 10 as a corollary of the existence of c independent cascade measures. The following representation for MID cascade generators is an analogue of the Levy–Khinchin representation:

$$\ln Ew_*^q = \int \ln E\pi_x^q \sigma(dx) \equiv \int (e^{-qx} - qe^{-x} + q - 1)\, \sigma(dx) \tag{5}$$

where $\ln \pi_x$ is a random Poisson quantity on a grid of step x, i.e., $P(\ln \pi_x = xn + b_x) = (n!e)^{-1}$, while b_x is specified by the normalization $E\pi_x = 1$. Here, $x^2\sigma(dx)$ is a local bounded measure on the line $(-\infty, \infty)$ with $\int (1 - (x+1)\, e^{-x})\, \sigma(dx) < 1$. The last requirement ensures the existence of a nontrivial cascade measure. For the derivation of (5) see ref. 12.

In ref. 11 the cascade models are treated in an extended manner using the operations of addition and multiplication for breakdown coefficients. We note in this connection that there exist random variables that are infinitely divisible both additively and multiplicatively at the same time. Such are, e.g., variables having lognormal or Gamma distributions; these were proposed for use in turbulent cascades of nonconservative type. In particular, the Gamma model[15] has the following spectral measure in (5):

$$\sigma_\alpha(dx) = e^{-x\alpha} x^{-1} (1 - e^{-x})^{-1}\, dx, \qquad x > 0, \quad \alpha > \alpha_0(c) > 0 \tag{6}$$

The spectral measure $\sigma(dx) = \sigma_\alpha(dx) - \sigma_\beta(dx)$, $0 < \alpha < \beta < \infty$, corresponds to a bounded variable $w_* = (\beta/\alpha)\, \xi_{\alpha, \beta-\alpha}$ of the MID type where $\xi_{a,b}$ has Euler's Beta distribution on the interval $[0, 1]$ with parameters (a, b). In ref. 6 this model with $\beta/\alpha = c$ is used to describe conservative turbulent cascades.

Thus, the class of scale invariant τ-functions is broad enough. We note nevertheless one defect in the mathematical analysis of independent cascades.

The convergence (3) has been proved only for c-adic divisions of the physical space where c is a branching parameter that is known exactly.

We now are going to examine the existence of c independent cascade measures.

3. CASCADE MEASURES THAT ARE INDEPENDENT OF THE BRANCHING PARAMETER

The following lemma is relevant to the analysis of mean cascade measures, $E\mu(dx)$. Below $\stackrel{d}{=}$ denotes equality in distribution.

Let $\xi_c = \sum_{i \geqslant 1} \varepsilon_i(c)\, c^{-i}$ where $\{\varepsilon_i(c)\}$ are independent, identically distributed random variables taking the values 0, 1,..., $c-1$.

Lemma. If $\xi_{c=2} \stackrel{d}{=} \xi_{c=3}$, then either the distribution of ξ_{c_i} is uniform or $\xi_{c_i} = 0$ or 1 a.s.

Proof. Let $Q_k(\eta)$ be the semi-invariant of the order k for a random variable η. Then

$$Q_k(\xi_c) = \sum_{i \geqslant 0} Q_k(\varepsilon_i(c)\, c^{-i}) = Q_k(\varepsilon(c))(c^k - 1)^{-1}$$

We will use the following relation between semi-invariants Q_k and moments m_k of a random variable:

$$Q_1 = m_1, \qquad Q_2 = m_2 - m_1^2, \qquad Q_3 = m_3 - 3m_1 m_2 + 2m_1^3$$

The moments $m_k(c)$ of $\varepsilon_i(c)$ are as follows: $m_k(2) = P(\varepsilon(2) = 1) := p$ and $m_k(3) = p_1 + 2^k p_2$, where $p_i = P(\varepsilon(3) = i)$. If $\xi_{c_1} \stackrel{d}{=} \xi_{c_2}$ with $c_1 = 2$ and $c_2 = 3$, then

$$Q_k(\varepsilon(3)) = (3^k - 1)(2^k - 1)^{-1}\, Q_k(\varepsilon(2))$$

Putting here $k = 1, 2, 3$, one gets three equations for the unknowns p, p_1, p_2:

$$p_1 + 2p_2 = 2p; \qquad p_1 + 4p_2 = 8/3 \cdot pq + 4p^2$$

$$p_1 + 8p_2 = 26/7 \cdot pq(1 - 2p) + 3(2p)(8/3 \cdot pq + 4p^2) - 2(2p)^3$$

Here, $q = 1 - p$. Adding up these equations with the weights 2, -3, 1, one gets an equation of degree three in p. It can be easily transformed to have the form $p(1-p)(1-2p) = 0$. The solution $p = 1/2$ leads to a uniform distribution of ξ_c, while $p = 0$ or $p = 1$ to δ-distributions concentrated at $x = 0$ and 1, respectively. The proof is complete. ∎

Corollary. If the cascade measure μ on $[0, 1]$ can be derived with branching parameters $c=2$ and 3, then either $\bar{\mu}(dx)=E\mu(x)$ is a Lebesgue measure or $\mu(dx)=\xi\delta(x-x_0)\,dx$, where $x_0=0$ or 1, $\xi>0$ and $E\xi=1$.

Proof. If μ is produced by the generator $\{w_i(c)\}$, then $\bar{\mu}(dx)$ is produced by $\bar{w}_i(c)=Ew_i(c)=p_i(c)$. This means that $\bar{\mu}(dx)$ is the distribution of the random variable ξ_c from the Lemma with $p_i(c)=P(\varepsilon(c)=i)$. According to the Lemma, $p_i(c)=1/c$ (Lebesgue measure) or $p_0(c)=1$, or else $p_{c-1}(c)=1$.

Statement. Consider cascade measures μ for which $P(\mu(\Delta)>0)=1$ on any subinterval $\Delta\in[0, 1]$ and $EM^2<\infty$, where $M=\mu[(0, 1)]$ is the total mass μ.

1. If a cascade measure is generated by conservative cascades with $c=2$ and 3, then μ is a Lebesgue measure;

2. If a cascade measure is generated by not necessarily conservative cascades with $c=2$, 3 and 5, then μ is a Lebesgue measure.

Proof. We begin by outlining the main idea of the proof. According to the Lemma, the mean measure $\bar{\mu}$ is a Lebesgue measure, i.e., $E\mu(\Delta)=|\Delta|$ and $Ew_i(c)=1/c$ for any measure generator $\{w_i(c)\}$. Relation (4) specifies how the generators are connected with various values of c. In particular,

$$\sum_0^{c-1} Ew_i^2(c)=\left[\sum_0^1 Ew_i^2(2)\right]^{\alpha_c}, \qquad \alpha_c=\ln c/\ln 2 \tag{7}$$

Since the generators $\{w_i(c)\}$ are not known, except for the conservative case, (7) should be modified as follows. Let $\Delta_i(c)$ be the division of $[0, 1]=I$ into c equal parts. In this case

$$\mu(\Delta_i(c))\overset{\mathrm{d}}{=}w_i(c)\,M_i, \qquad i=1,\ldots, c$$

where M_i are independent copies of the total mass M of the measure μ and are independent of $\{w_i(c)\}$. Hence (7) can be replaced by

$$\sum_{i=0}^{c-1} E\mu^2(\Delta_i(c))=(x+y)^{\alpha_c}\,m_2 \tag{8}$$

where $x=Ew_0^2(2)$, $y=Ew_1^2(2)$ and $m_2=EM^2$.

The measure μ is fully specified by the distribution of the generator $\{w_0(2), w_1(2)\}$. For this reason $E\mu^2(\Delta_i(c))$ can be expressed in terms of the first and second moments of $\{w_\alpha(2)\}$. The first moments of $\{w_\alpha(2)\}$ are known from the Lemma: $Ew_\alpha(c) = 1/c$. The remaining ones are x, y, $\rho = Ew_0(2)\, w_1(2)$, and m_2. From the equality

$$M \overset{\mathrm{d}}{=} \sum_{i=0}^{c-1} w_i(c)\, M_i$$

where $\{w_i(c)\}$ and $\{M_i\}$ are independent, $M_i \overset{\mathrm{d}}{=} M$, we get a relation between ρ and m_2:

$$m_2(1-x-y) = 2\rho \tag{9}$$

The relation can be used to eliminate the unknown ρ. It turns out that $E\mu^2(\Delta_i(c))$ is proportional to m_2. As a result, Eq. (8) actually contains two unknowns only, x and y.

If the cascade is conservative, then $x = y$. This can be seen as follows. The mean values of $w_0(2)$ and $w_1(2)$ are equal, and so are the variances, because $w_0(2) + w_1(2) = 1$. Therefore the second moments of $w_\alpha(2)$ are equal too, i.e., $x = y$. This means that it is sufficient to have a single equation like (8) for $c = 3$ in the conservative case, and two equations for $c = 3$ and $c = 5$ in the nonconservative case.

Thus, one has to find a stochastic representation of $\mu(\Delta_i(c))$ in terms of independent copies of the generator $(w_0(2), w_1(2))$. Let Δ_n be a dyadic subinterval of (0, 1) with length 2^{-n}. We will denote it by the word $A(\Delta_n) = \sigma_1,..., \sigma_n$, $\sigma_i = 0$ or 1, if $0 \cdot \sigma_1 \cdots \sigma_n$ is the left endpoint of the interval Δ_n. It follows from the definition of the independent cascade that

$$\mu(\Delta_n) \overset{\mathrm{d}}{=} w_{\sigma_1} w_{\sigma_1\sigma_2} \cdots w_{\sigma_1,..., \sigma_n} M_{\sigma_1,..., \sigma_n} := \xi_A M_A \tag{10}$$

where $w_{\sigma_1,..., \sigma_n}$ are independent variables for different n, and $(w_{\sigma_1,..., \sigma_{n-1}, 0}, w_{\sigma_1,..., \sigma_{n-1}, 1}) \overset{\mathrm{d}}{=} (w_0(2), w_1(2))$; the M_A are independent for different words A and $M_A \overset{\mathrm{d}}{=} M$, M_A and $w_{A'}$ being mutually independent.

The joint representation of $\mu(\Delta_{n_1})$ and $\mu(\Delta_{n_2})$ can be derived as follows. Let $A = A(\Delta_{n_1}) = A_0 A_1$ and $B = A(\Delta_{n_2}) = A_0 A_2$, where A_0 is the longest word that is common to A and B; A_1 and A_2 are extensions of the word A_0 in A and B, respectively. Hence

$$\mu(\Delta_{n_1}) = \xi_{A_0} \xi_{A_1} M_A, \qquad \mu(\Delta_{n_2}) = \xi_{A_0} \xi_{A_2} M_B \tag{11}$$

where the structure of ξ_A has been described above (see (10)) and ξ_{A_0}, ξ_{A_1}, ξ_{A_2}, M_A, M_B are independent.

The measure μ is σ-additive a.s. Consequently, partitioning an arbitrary interval Δ into nonoverlapping dyadic intervals, one gets a stochastic representation of $\mu(\Delta_i(c))$. Consider the case $c = 3$ in more detail.

The Case c=3. Let (w_0^i, w_1^i) and $(\tilde{w}_0^i, \tilde{w}_1^i)$, $i=1,2$ be independent copies of the generator $\{w_0(2), w_1(2)\}$; also, $\{M^i\}$ are independent copies of M and are independent of $\{w_\alpha^i, \tilde{w}_\alpha^i\}$. Consider the intervals $\Delta_i = \Delta_i(c=3)$.

The interval $\Delta_0 = (0, 1/3)$; $1/3 = 0.(01)$ in binary form gives information on the dyadic structure of Δ_0. Taking into account (10, 11), we can write a stochastic equation for $\mu(\Delta_0)$:

$$\mu(\Delta_0) \overset{d}{=} w_0^1 w_0^2 M^1 + w_0^1 w_1^2 T^2[\mu(\Delta_0)] \tag{12}$$

where $\mu(\Delta_0)$ is to be found in the form of a functional of $\{w_\alpha^i, M^i, i=1,2,...\}$, T is the translation over the superscripts: $Tw_\alpha^i = w_\alpha^{i+1}$, $TM^i = M^{i+1}$. The desired representation of $\mu(\Delta_0)$ is obtained by iterations of (12) starting from $\mu(\Delta_0) = 0$. Convergence a.s. is guaranteed by general cascade results (see ref. 5). From (12) one gets

$$E\mu^2(\Delta_0) = x^2 m_2 + 2x\rho EM^1 E\mu(\Delta_0) + xyE\mu^2(\Delta_0)$$

Here the independence of M^1 and $T^2[w_\alpha^i, M_\alpha^i, i=1,2,...]$ has been used. The substitutions $EM^1 = 1$, $E\mu(\Delta_0) = |\Delta_0| = 1/3$ and $2\rho = (1-x-y)\, m_2$ yield

$$E\mu^2(\Delta_0) = (x^2 + (1-z)/3)(1-v)^{-1}\, m_2 \tag{13}$$

where $z = x+y$, $v = xy$.

The interval $\Delta_2 = (2/3, 1)$ is obtained from Δ_0 by reflection transformation: if $\Delta = (a, b)$, then $\Delta^c = (1-b, 1-a)$, i.e., $\Delta_2 = \Delta_0^c$. We now define an operation $[\ ]^c$ to act on the basis w_α^i : $[w_\alpha^i]^c = w_{\alpha^c}^i$, where $\alpha^c = 1-\alpha$ with $\alpha = 0$ and 1. It is easy to see that, once a representation of $\mu(\Delta)$ through the basis $\{w_\alpha^i\}$ has been found, then $\mu(\Delta^c) \overset{d}{=} [\mu(\Delta)]^c$. Recalling that the basis elements w_α^i are independent and the means $Ew_\alpha^i = 1/2$ are homogeneous, one gets $E\mu^2(\Delta^c) = \{E\mu^2(\Delta)\}^c$, where $\{\cdot\}^c$ means that the unknowns (x, y) are interchanged: $(x, y) \to (y, x)$. For this reason, knowing (13), one gets

$$E\mu^2(\Delta_2) = \{E\mu^2(\Delta_0)\}^c = (y^2 + (1-z)/3)(1-v)^{-1}\, m_2 \tag{14}$$

It remains to find $\mu(\Delta_1)$. One has $\Delta_1 = (1/3, 1/2) \cup (1/2, 2/3) := \delta^c \cup \delta$, where $\delta = 1/2 + 1/2\Delta_0$. Taking into account the above formalism, one has

$$\mu(\Delta_1) \overset{d}{=} M_0^1 T[\mu(\Delta_0)]^c + M_1^1 T[\tilde{\mu}(\Delta_0)]$$

where $\tilde{\mu}(\Delta_0)$ is an independent copy of $\mu(\Delta_0)$ based on the basis $\{\tilde{w}^i_\alpha \tilde{M}^i\}$. Hence

$$E\mu^2(\Delta_1) = x\{E\mu^2(\Delta_0)\}^c + yE\mu^2(\Delta_0) + 2\rho[E\mu(\Delta_0)]^2 \tag{15}$$

It remains to substitute (13), $E\mu(\Delta_0) = 1/3$, and $2\rho = 1 - z$ in this equation. Combining (13)–(15), one gets the first equation of type (7) after some simple manipulations:

$$(1-z)[2/3 \cdot (z+2)(1-v)^{-1} - 4/9] = 1 - z^{\alpha_3}, \qquad \alpha_3 = \ln 3/\ln 2 \tag{16}$$

where $z = x + y$, $v = xy$.

The Solution $x+y=1$. From (16) it follows that (16) has one obvious family of solutions $(x, y) : x + y = 1$. Recalling (9), that will mean the following:

$$E(w_0 + w_1)^2 = 1 = E(w_0 + w_1), \qquad Ew_0 w_1 = 0, \quad w_i \geqslant 0$$

This is possible, if $w_0 + w_1 = 1$. However, $w_0 w_1 = 0$ and $Ew_i = 1/2$, hence $P(w_0 = 0, w_1 = 1) = P(w_0 = 1, w_1 = 0) = 1/2$. Such a generator produces the measure $\mu(dx) = \delta(x - \xi)\, dx$, where ξ is a random variable with uniform distribution on $[0, 1]$. This type of measure is ruled out by the Statement.

Conservative Cascades. If $x = y$, Eq. (16) can be reduced to the following:

$$4/9 \cdot (2+x)(1-x)^{-1} = (1-(2x)^{\alpha_3})(1-2x)^{-1}, \qquad 0 < x \leqslant 1$$

We have taken into account the fact that $z = 2x \neq 1$. It is easy to see that the equation has a single root $x = 1/4$, i.e., $Ew_\alpha^2(2) = [Ew_\alpha(2)]^2$. Consequently, the variance of $w_\alpha(2)$ is zero, and $w_0(2) = w_1(2) = 1/2$. Such a generator produces the Lebesgue measure. The first part of the Statement is proven.

The Case $c=5$. The interval is $\Delta_0(c) = (0, 1/5)$, where $1/5 = 0.(0011)$ in binary form. The analogue of (12) is

$$\mu(\Delta_0) \stackrel{d}{=} w_0^1 w_0^2 w_0^3 M^1 + w_0^1 w_0^2 w_1^3 w_0^4 M^2 + w_0^1 w_0^2 w_1^3 w_1^4 T^4[\mu(\Delta_0)]$$

Hence

$$\begin{aligned} E\mu^2(\Delta_0) = (&x^3 m_2 + x^3 y m_2 + 2x^2 \rho (EM)^2\, Ew_0 \\ &+ 2x^2 \rho EM E\mu(\Delta_0) + 2x^2 y \rho EM E\mu(\Delta_0)) \times (1 - x^2 y^2)^{-1} \end{aligned}$$

Recall again that $EM=1$, $Ew_0=1/2$, $E\mu(\Delta_0)=1/5$ and $2\rho=(1-x-y)\,m_2$. One has

$$E\mu^2(\Delta_0)=x^2(x+v+(1-z)(3+y)/5)(1-v^2)^{-1}\,m_2 \tag{17}$$

The interval is $\Delta_1=(1/5,2/5)=(1/5,1/4)\cup(1/4,3/8)\cup(3/8,2/5)=1/4\Delta_0^c\cup(1/4,3/8)\cup(3/8+1/8\,\Delta_0)$. Consequently,

$$\mu(\Delta_1)\overset{\mathrm{d}}{=}w_0^1w_0^2T^2[\mu(\Delta_0)]^c+w_0^1w_1^2\tilde{w}_0^3\tilde{M}^1+w_0^1w_1^2\tilde{w}_0^3T^3[\tilde{\mu}(\Delta_0)]$$

where $\tilde{\mu}(\Delta_0)$ is an independent copy of $\mu(\Delta_0)$ based on the basis $\{\tilde{w}_\alpha^i,\tilde{M}^i\}$. Consequently,

$$\begin{aligned}E\mu^2(\Delta_1)=x^2\{E\mu^2(\Delta_0)\}^c+x^2ym_2+xy^2E\mu^2(\Delta_0)\\+2x(1-z)(3+5x)\,5^{-2}m_2\end{aligned} \tag{18}$$

The fourth term is given here as transformed by using (9).

The interval is $\Delta_2=(2/5,3/5)=1/2\,\Delta_0^c\cup(1/2+1/2\,\Delta_0)$. Therefore,

$$\mu(\Delta_2)\overset{\mathrm{d}}{=}w_0^1T[\mu(\Delta_0)]^c+w_0^1T[\tilde{\mu}(\Delta_0)]$$

so that

$$E\mu^2(\Delta_2)=x\{E\mu^2(\Delta_0)\}^c+yE\mu^2(\Delta_0)+2(1-z)\,5^{-2}m_2 \tag{19}$$

The intervals are $\Delta_3=(3/5,4/5)=\Delta_1^c$, $\Delta_4=(4/5,1)=\Delta_0^c$. Therefore,

$$E\mu^2(\Delta_3)=\{E\mu^2(\Delta_1)\}^c,\qquad E\mu^2(\Delta_4)=\{E\mu^2(\Delta_0)\}^c \tag{20}$$

Substitute (17)–(20) into (8). After some simple algebra that needs some care however, one gets the second desired equation $\Phi=0$ where

$$\Phi(z,v):=\frac{z^2+4}{1-v^2}+\frac{3z+2}{1-v}-0.5z^2-0.8z+2v-3.6-2.5\,\frac{1-z^{\alpha_5}}{1-z} \tag{21}$$

Here $z=x+y$, $v=x\cdot y$, $\alpha_5=\ln 5/\ln 2$.

Equation (16) can be used to find an explicit expression of v in terms of z:

$$1-v=2/3(z+2)[4/9+(1-z^{\alpha_3})/(1-z)]^{-1} \tag{22}$$

Substitution of (22) into (21) yields an equation $\Phi(z,v(z))=0$ for z. The relation

$$E\sum_{i=1}^{c-1}w_i^2(c)=c^{-\tau(2)}$$

and the inequality $\tau(2) > \tau(1) = 0$ mean that $z = x + y \leqslant 1$. Numeric calculations show that the equation $\Phi(z, v(z)) = 0$ has the single root $z = 1/2$ in the interval (0, 1). But then, $v(z = 1/2) = 1/16$ and $x = y = 1/4$. As has been shown above, this solution leads to the Lebesgue measure. The proof is complete.

4. CONCLUSION

The above result roughly means that the multifractal cascade measure remembers the dimension of its generator. Measures that have no such property are scale invariant and lose the property of intermittency. Measures that are independent of the branching parameter emerge, explicitly or implicitly, to substantiate the multiplicative infinite divisibility[9, 10, 16] or universality[11] of cascade generators. The statistical conclusions about interscale dependence of breakdown coefficients in turbulent cascades[6, 16] were essentially based on the knowledge of the branching parameter c. In other words, independent cascade models become rather restrictive, when one tries to get beyond merely qualitative explanations of intermittency. For this reason the Parisi-Frisch explanation of the intermittency phenomenon based on the assumption of multifractality for physical fields seems more flexible. The explanation leaves aside the generating mechanism of multifractality, which can result from different origins and is capable of different formalizations. An example is provided by simple sedimentation models in geology where multifractality not produce by a multiplicative process.[17]

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